

Optimal Radix-2 FFT Compatible Filters for GFDM

Ahmad Nimr, Maximilian Matthé, Dan Zhang, Gerhard Fettweis

Vodafone Chair Mobile Communication Systems, Technische Universität Dresden, Germany

{first name.last name}@ifn.et.tu-dresden.de

Abstract—For a linear waveform, a finite condition number of the corresponding modulation matrix is necessary for the waveform to convey the message without ambiguity. Based on the Zak transform, this letter presents an analytical approach to compute the condition number of the modulation matrix for the multi-carrier waveform generalized frequency division multiplexing (GFDM). On top, we further propose a filter design that yields non-singular modulation matrices for an even number of subcarriers and subsymbols, which is not achievable for any previous work. Such new design has significant impact on implementation complexity, as the radix-2 FFT operations for conventional multicarrier waveforms can readily be employed for GFDM. Additionally, we analytically derive the optimal filter that minimizes the condition number. We further numerically evaluate the signal-to-interference ratio (SIR) and noise-enhancement factor (NEF) for matched filter (MF) and zero-forcing (ZF) GFDM receivers for such design, respectively.

Index Terms—GFDM, Zak transform, pulse shape, conditional number, even number of subsymbols.

I. INTRODUCTION

Among several waveform alternatives to orthogonal frequency division multiplexing (OFDM) [1], considerable research on detection algorithms, performance and low-complexity implementations has been conducted for GFDM [2]. GFDM, as a non-orthogonal filtered multicarrier system with K subcarriers employs circular filtering of M subsymbols within each block to keep the signal confined within the block duration of KM samples. Naturally, the choice of the pulse shaping filter strongly influences the system performance, as it controls system orthogonality and interference structure. In [3], it is proved by means of the discrete Zak transform (DZT) [4] of the transmit filter that the transmit signal becomes ambiguous and the GFDM modulation matrix $\mathbf{A}$ is singular when a real-valued symmetric filter with even M and K is employed. The authors of [5] extended this result by showing that $\mathbf{A}$ has exactly one zero eigenvalue, suggesting that one GFDM block can mostly convey $(KM - 1)$ data symbols. Hence, odd M is commonly adopted for data transmission in the literature. Even though several works on complexity reduction for GFDM modulation and demodulation have been published [6], [7], odd M forbids the N -point FFT to be implemented solely by energy-efficient radix-2 based processing. This also narrows the design space of GFDM as a flexible waveform generator [8].

In this paper we propose a filter design for GFDM to supports even values for both M and K , particularly when they are power-of-two. To this end, we introduce a fractional shift in the sampling of the continuous frequency response of conventional basis filters, such as raised-cosine (RC) filter, to allow both even-valued and odd-valued M, K to be derived

from the same filter response. As a function of this shift, a closed-form expression for the condition number of $\mathbf{A}$ is provided and the optimal shift for both even and odd M, K in terms of the minimal condition number is derived. To verify the design we evaluate the SIR for the MF receiver and the NEF of the ZF receiver.

II. GFDM MODULATION MATRIX DECOMPOSITION

One GFDM block conveys the data symbols $\{d_{k,m}\}$ via K subcarriers and M subsymbols, yielding $N = KM$ samples. The n th one as the entry n of $\mathbf{x} \in \mathbb{C}^{N \times 1}$ equals [2]

$$[\mathbf{x}]_n = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} d_{k,m} g[\langle n - mK \rangle_N] e^{j2\pi \frac{k}{K} n}, \quad (1)$$

where $g[n]$ denotes the pulse shaping filter and corresponds to the entry n of $\mathbf{g} \in \mathbb{C}^{N \times 1}$. With the $N \times N$ modulation matrix $\mathbf{A}$ constructed as $[\mathbf{A}]_{(n,k+mK)} = g[\langle n - mK \rangle_N] e^{j2\pi \frac{k}{K} n}$, Eq. (1) can be formed as $\mathbf{x} = \mathbf{A}\mathbf{d}$, where $[\mathbf{d}]_{k+mK} = d_{k,m}$.

A. Decomposition of $\mathbf{A}$

For a $L \times Q$ matrix $\mathbf{X}$, let $\mathbf{x} = \text{vec}_{L,Q}(\mathbf{X})$ and $\text{unvec}_{L,Q}(\mathbf{x})$ denote the vectorization operation and its inverse. Let $\mathbf{F}_N$ be the N -point discrete Fourier transform (DFT) matrix with elements $[\mathbf{F}_N]_{(i,j)} = e^{-j2\pi \frac{ij}{N}}$. Let the unitary matrix $\mathbf{U}_{L,Q}$ be

$$\mathbf{U}_{L,Q} = \frac{1}{\sqrt{Q}} \mathbf{I}_L \otimes \mathbf{F}_Q, \quad (2)$$

where $\otimes$ is the Kronecker product. Let $\mathbf{\Pi}_{L,Q} \in \mathbb{R}^{LQ \times LQ}$ be the permutation matrix that fulfills for any $L \times Q$ matrix $\mathbf{X}$

$$\text{vec}(\mathbf{X}^T) = \mathbf{\Pi}_{L,Q} \text{vec}(\mathbf{X}). \quad (3)$$

The (Q, L) DZT [4] $\mathbf{Z}\mathbf{x} = (\mathbf{F}_Q \otimes \mathbf{I}_L) \mathbf{x}$, $\mathbf{x} \in \mathbb{C}^{QL \times 1}$ can be written as a matrix $\mathbf{Z}_{Q,L}^{(\mathbf{x})} \in \mathbb{C}^{Q \times L}$ with

$$\mathbf{Z}_{Q,L}^{(\mathbf{x})} = \mathbf{F}_Q \mathbf{V}_{Q,L}^{(\mathbf{x})} = \tilde{\mathbf{V}}_{Q,L}^{(\mathbf{x})}, \quad (4)$$

$$\text{where } \mathbf{V}_{Q,L}^{(\mathbf{x})} = (\text{unvec}_{L \times Q} \{\mathbf{x}\})^T. \quad (5)$$

Resorting to the DZT of $\mathbf{g}$ and $\tilde{\mathbf{g}} = \mathbf{F}_N \mathbf{g}$, we can factorize $\mathbf{A}$ into two forms

$$\mathbf{A} = \underbrace{\mathbf{\Pi}_{K,M}^T \mathbf{U}_{K,M}^H}_{\mathbf{U}^{(g)}} \underbrace{\mathbf{\Lambda}^{(g)} \mathbf{U}_{K,M} \mathbf{\Pi}_{K,M} \mathbf{U}_{M,K}^H}_{\mathbf{V}^{(g)H}}, \quad (6)$$

$$= \underbrace{\frac{\mathbf{F}_N^H}{\sqrt{N}} \mathbf{\Pi}_{M,K}^T \mathbf{U}_{M,K}^H}_{\mathbf{U}^{(\tilde{g})}} \underbrace{\mathbf{\Lambda}^{(\tilde{g})} \mathbf{U}_{M,K} \mathbf{\Pi}_{M,K} \mathbf{U}_{K,M} \mathbf{\Pi}_{M,K}}_{\mathbf{V}^{(\tilde{g})H}}, \quad (7)$$

where $\mathbf{U}^{(\tilde{g})}$ and $\mathbf{V}^{(\tilde{g})}$ are unitary matrices and

$$\mathbf{\Lambda}^{(g)} = \text{diag} \left\{ \text{vec}_{M,K} \left\{ \sqrt{K} \mathbf{Z}_{M,K}^{(g)} \right\} \right\}, \quad (8)$$

$$\mathbf{\Lambda}^{(\tilde{g})} = \text{diag} \left\{ \text{vec}_{M,K} \left\{ \frac{1}{\sqrt{K}} \mathbf{Z}_{K,M}^{(\tilde{g})} \right\} \right\}, \quad (9)$$

where $\mathbf{\Lambda}^{(g)}$ and $\mathbf{\Lambda}^{(\tilde{g})}$ contain the DZT of $\mathbf{g}$ and $\tilde{\mathbf{g}}$, respectively. As a result, properties of $\mathbf{A}$ are dictated by $\mathbf{\Lambda}^{(\tilde{g})}$ or $\mathbf{\Lambda}^{(g)}$. In this paper we focus on pulse shapes that are sparse in frequency, hence we employ $\mathbf{\Lambda}^{(\tilde{g})}$ for the subsequent analysis.

B. Performance indicators

Define the short-hand notation $z_{k,m} = \left[\mathbf{Z}_{K,M}^{(\tilde{g})} \right]_{(k,m)}$. Then $\sigma_{k,m}^2 = |z_{k,m}|^2$ correspond to the squared singular values of $\mathbf{A}$ scaled by K . The conditional number of $\mathbf{A}$ is given by [9]

$$\text{cond}(\mathbf{A}) = \frac{\max_{k,m} \{\sigma_{k,m}\}}{\min_{k,m} \{\sigma_{k,m}\}} = \frac{\sigma_{\max}}{\sigma_{\min}}. \quad (10)$$

Considering the received signal in AWGN channel [10] the NEF of ZF and the SIR of the MF receiver can be written as

$$\begin{aligned} \text{NEF} &= \frac{1}{N^2} \|\mathbf{A}\|_F^2 \|\mathbf{A}^{-1}\|_F^2 = \frac{1}{N^2} \left\| \mathbf{\Lambda}^{(\tilde{g})} \right\|_F^2 \left\| \mathbf{\Lambda}^{(\tilde{g})^{-1}} \right\|_F^2 \\ &= \frac{1}{N^2} \left(\sum_{k,m} \sigma_{k,m}^2 \right) \left(\sum_{k,m} \frac{1}{\sigma_{k,m}^2} \right), \end{aligned} \quad (11)$$

$$\begin{aligned} \text{SIR} &= \frac{1}{N} \left\| \frac{\mathbf{A}^H \mathbf{A}}{\|\mathbf{g}\|^2} - \mathbf{I}_N \right\|_F^2 = \frac{1}{N} \left\| \frac{\mathbf{\Lambda}^{(\tilde{g})} \mathbf{\Lambda}^{(\tilde{g})H}}{\|\mathbf{g}\|^2} - \mathbf{I}_N \right\|_F^2 \\ &= \frac{1}{N} \sum_{k,m} \left(\frac{\sigma_{k,m}^2}{\sum_{k,m} \sigma_{k,m}^2} - 1 \right)^2. \end{aligned} \quad (12)$$

III. GFDM PULSE SHAPING FILTER DESIGN

The conventional pulse shaping filter design for the GFDM is to let $\mathbf{g} = \mathbf{F}_N^H \tilde{\mathbf{g}}$ with $[\tilde{\mathbf{g}}]_n = H(\frac{n}{N})$, where $H(\nu)$ stands for the discrete-time Fourier transform (DTFT) of a pre-selected basis filter $h[n]$ that is of practical interests, e.g., RC or Root-Raised Cosine (RRC). Here, ν is the normalized frequency and thus the period of $H(\nu)$ is equal to 1. With such design of $\mathbf{g}$, it has been shown in [3] that $\mathbf{A}$ becomes singular for even M, K and a real symmetric filter $h[n]$. This is caused by $\left[\mathbf{Z}_{K,M}^{(\tilde{g})} \right]_{(k,m)} = (\mathcal{Z}H(\nu))(\frac{k}{K}, \frac{m}{M})$, where $(\mathcal{Z}H(\nu))(f, t)$ denotes the discrete-time Zak transform of $H(\nu)$ and for any real symmetric filter we know $(\mathcal{Z}H(\nu))(\frac{1}{2}, \frac{1}{2}) = 0$. The requirement of odd M or K impedes an efficient implementation in terms of low-complexity radix-2 FFT operations. In the sequel, we propose a novel design approach that overcomes this restriction for any basis filter $h[n]$ fulfilling the following conditions

- 1) $h[n]$ is real-valued, i.e. $H(\nu) = H^*(1 - \nu) = H^*(-\nu)$.
- 2) $H(\nu)$ spans two subcarriers within each period, i.e. $H(\nu) = 0, \forall \nu \in [\frac{1}{K}, \frac{1}{2}]$.
- 3) $|H(\nu)|$ is decreasing from 1 to 0 for $\nu \in [0, \frac{1}{K}]$.

We start from noting

$$|(\mathcal{Z}H(\nu - \eta))(f, t)| = |(\mathcal{Z}H(\nu))(f - \eta, t)|, \quad (13)$$

namely, shifting the frequency response of a filter also shifts the frequency coordinate of its Zak transform [4]. Hence, shifting $H(\nu)$ can help us avoid to sample the zero in $(\mathcal{Z}H(\nu))$ for even M, K . Accordingly, the samples of $\tilde{\mathbf{g}}$ are defined as

$$[\tilde{\mathbf{g}}]_n(\lambda) = \begin{cases} H\left(\frac{n+\lambda}{N}\right), & 0 \leq n < M - \lambda \\ H^*\left(\frac{N-n-\lambda}{N}\right), & N - M - \lambda < n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \quad (14)$$

for $\lambda \in [0, 1[$. $\tilde{\mathbf{g}}$ can be reshaped as in (5) to

$$\left[\mathbf{V}_{K,M}^{(\tilde{g})}(\lambda) \right]_{(k,m)} = \begin{cases} H\left(\frac{m+\lambda}{N}\right), & k = 0 \\ H^*\left(\frac{M-m-\lambda}{N}\right), & k = K - 1 \\ 0, & \text{elsewhere} \end{cases}. \quad (15)$$

Applying DFT according to (4), we get

$$z_{k,m}(\lambda) = H\left(\frac{m+\lambda}{N}\right) + H^*\left(\frac{M-m-\lambda}{N}\right) e^{j2\pi \frac{k}{K}}. \quad (16)$$

Due to the symmetry of $H(\nu)$, we have

$$\begin{aligned} z_{k,m}(1 - \lambda) &= z_{k,M-1-m}^*(\lambda) e^{j2\pi \frac{k}{K}}, \\ \sigma_{k,m}^2(1 - \lambda) &= \sigma_{k,M-1-m}^2(\lambda). \end{aligned} \quad (17)$$

Hence, all results regarding conditional number, NEF and SIR are symmetric around $\lambda = 0.5$. Moreover, Eq. (16) shows that $z_{k,m}(1 + \lambda) = z_{k,m+1}(\lambda)$. Hence, it suffices to study the case $0 \leq \lambda \leq 0.5$. Additionally, we focus on $K = 2^x$ for $x > 1$.

To obtain closed-forms of the condition number of $\mathbf{A}$, we subsequently focus on two particular families of $H(\nu)$, namely well-localized filters that fulfill the inter-symbol-interference (ISI)-free criterion without or with matched filtering.

A. ISI free without matched filter

In this case $H(\nu)$ additionally satisfies

$$\sum_{k=0}^{K-1} H\left(\nu - \frac{k}{K}\right) = 1. \quad (18)$$

From the symmetry and limited band of $H(\nu)$ it follows

$$H(\nu) + H^*\left(\frac{1}{K} - \nu\right) = 1, \quad \forall \nu \in [0, \frac{1}{K}] \quad (19)$$

and $H\left(\frac{m+\lambda}{N}\right) + H^*\left(\frac{M-m-\lambda}{N}\right) = 1$. Also, there exists a function $f(\nu) = r(\nu) e^{j\phi(\nu)}$ with $f(\nu) = -f^*\left(\frac{1}{K} - \nu\right)$ and

$$H(\nu) = \frac{1}{2} (1 + f(\nu)), \quad \forall \nu \in [0, \frac{1}{K}]. \quad (20)$$

Let us assume a real-valued $f(\nu)$, i.e. $\phi(\nu) = 0$ and $f(\nu) = r(\nu)^1$. Due to the constraint of decreasing amplitude $H(\nu)$, $r(\nu)$ must be decreasing from 1 to -1 for $\nu \in [0, \frac{1}{K}]$. Based on (19) and (20) we get

$$\sigma_{A_{k,m}}^2(\lambda) = \frac{(1 + f_m^2(\lambda))}{2} + \frac{(1 - f_m^2(\lambda))}{2} \cos\left(2\pi \frac{k}{K}\right), \quad (21)$$

¹Complex $f(\nu)$ as in Xia-filters [11] is treated in the following section.

where $f_m(\lambda) = f(\frac{m+\lambda}{N}) = 2H(\frac{m+\lambda}{N}) - 1$. The singular values are symmetric with respect to k , and decreasing with $k = 0, \dots, \frac{K}{2}$. Therefore, $\sigma_{A_{0,m}}^2(\lambda) = 1$ and $\sigma_{A_{\frac{K}{2},m}}^2(\lambda) = f_m^2(\lambda)$ are the maximum and minimum singular value with respect to k , respectively. Therefore, $\sigma_{A_{max}}^2(\lambda) = 1$, because $f_m^2(\lambda) \leq 1$, and $\sigma_{A_{min}}^2(\lambda)$ is obtained from $\min_m \{f_m(\lambda)^2\}$. Since $f(\nu)$ is decreasing and antisymmetric around $\frac{1}{2K}$, $f(\nu)^2$ is decreasing $\forall \nu \in [0, \frac{1}{2K}]$ and increasing $\forall \nu \in [\frac{1}{2K}, \frac{1}{K}]$. As a result, when M is even, $0 \leq \lambda \leq 0.5$, $\sigma_{A_{min}}^2$ is obtained at $m = M/2$, and when M is odd, it is obtained at $m = (M-1)/2$. Therefore,

$$\sigma_{A_{min}}^2(\lambda) = f^2\left(\frac{1}{2K} + \frac{S(\lambda)}{2N}\right). \quad (22)$$

$$\text{where } S(\lambda) = \begin{cases} 2\lambda, & M \text{ is even} \\ 1-2\lambda, & M \text{ is odd} \end{cases}. \quad (23)$$

From the increasing/decreasing intervals of $f^2(\nu)$, $\sigma_{A_{min}}^2(\lambda)$ increases with $0 \leq \lambda \leq 0.5$ for even M and decreases when M is odd. Hence, the condition number can be expressed as

$$\text{cond}(\mathbf{A}_A)(\lambda) = \frac{1}{\left|f\left(\frac{1}{2K} + \frac{S(\lambda)}{2N}\right)\right|}. \quad (24)$$

Similarly, $\text{cond}(\mathbf{A}_A)(\lambda)$ is decreasing for even M and increasing for odd M . Hence, the best condition of $\mathbf{A}$ is attained at $\lambda = 0.5$ for even M and $\lambda = 0$ for odd M .

B. ISI free after matched filtering

A filter $H(\nu)$ is ISI-free after matched filtering if

$$\sum_{k=0}^{K-1} \left|H\left(\nu - \frac{k}{K}\right)\right|^2 = 1. \quad (25)$$

By exploiting the symmetry and band limit, we get

$$|H(\nu)|^2 + \left|H^*\left(\frac{1}{K} - \nu\right)\right|^2 = 1, \quad \forall \nu \in [0, \frac{1}{K}], \quad (26)$$

and hence $|H(\frac{m+\lambda}{N})|^2 + |H^*(\frac{M-m-\lambda}{N})|^2 = 1$. Furthermore, there exists a real-valued function $f(\nu) = -f(\frac{1}{K} - \nu)$, which is decreasing from 1 to -1 in the interval $\nu \in [0, \frac{1}{K}]$ with

$$|H(\nu)|^2 = \frac{1}{2}(1 + f(\nu)), \quad \forall \nu \in [0, \frac{1}{K}]. \quad (27)$$

Adding an (arbitrary) phase $\phi(\nu)$ yields the original $H(\nu)$ by

$$H(\nu) = e^{j\phi(\nu)} \sqrt{\frac{1}{2}(1 + f(\nu))}, \quad \forall \nu \in [0, \frac{1}{K}]. \quad (28)$$

Using (26) and (28),

$$\begin{aligned} H\left(\frac{m+\lambda}{N}\right) &= e^{j\phi_{a,m}(\lambda)} \sqrt{\frac{1}{2}(1 + f_m(\lambda))}, \\ H^*\left(\frac{M-m-\lambda}{N}\right) &= e^{j\phi_{b,m}(\lambda)} \sqrt{\frac{1}{2}(1 - f_m(\lambda))}. \end{aligned} \quad (29)$$

where $\phi_m^a(\lambda) = \phi(\frac{m+\lambda}{N})$, $\phi_m^b(\lambda) = -\phi(\frac{M-m-\lambda}{N})$, and $f_m(\lambda) = f(\frac{m+\lambda}{N})$. As special cases, we study the phase in the form $\phi(\nu) = -\phi(\frac{1}{K} - \nu) + \beta\frac{\pi}{2}$, $\beta = 0, 1, 2, 3$. Then $e^{j\phi_{a,m}(\lambda)} = j^\beta e^{j\phi_{b,m}(\lambda)}$. No ISI with and without MF, as the

Xia filters [11] provide, is obtained with $f(\nu) = \cos(2\phi(\nu))$ and $\beta = 2$ or, equally, $\phi(\nu) = \frac{1}{2}\arccos(f(\nu))$. From (16), we get

$$\sigma_{B_{k,m}}^2(\lambda) = 1 + \sqrt{1 - f_m^2(\lambda)} \cos\left(2\pi \frac{k - \beta \frac{K}{4}}{K}\right). \quad (30)$$

The maximum singular value with respect to k is located at $k_{max} = \beta \frac{K}{4}$ and the minimum one at $k_{min} = (\beta + 2 \bmod 4) \frac{K}{4}$. This requires that K is a multiple of 4 for $\beta = 1, 3$.

$$\begin{aligned} \sigma_{B_{k_{max},m}}^2(\lambda) &= 1 + \sqrt{1 - f_m^2(\lambda)}, \\ \sigma_{B_{k_{min},m}}^2(\lambda) &= 1 - \sqrt{1 - f_m^2(\lambda)}. \end{aligned} \quad (31)$$

Following the same argument as previously, based on the properties of $f(\nu)$, both $\sigma_{B_{min}}^2(\lambda)$ and $\sigma_{B_{max}}^2(\lambda)$ are obtained at $m = M/2$ for even M and $m = \frac{M-1}{2}$ and for odd M . Thus,

$$\begin{aligned} \sigma_{B_{max}}^2(\lambda) &= 1 + \sqrt{1 - f^2\left(\frac{1}{2K} + \frac{S(\lambda)}{2N}\right)}, \\ \sigma_{B_{min}}^2(\lambda) &= 1 - \sqrt{1 - f^2\left(\frac{1}{2K} + \frac{S(\lambda)}{2N}\right)}, \end{aligned} \quad (32)$$

and the conditional number can then be written as

$$\text{cond}(\mathbf{A}_B)(\lambda) = \frac{\left|f\left(\frac{1}{2K} + \frac{S(\lambda)}{2N}\right)\right|}{1 - \sqrt{1 - f^2\left(\frac{1}{2K} + \frac{S(\lambda)}{2N}\right)}}. \quad (33)$$

$\text{cond}(\mathbf{A}_B)(\lambda)$ is decreasing for even M and increasing for odd M with $\lambda \in [0, 0.5]$. When using the same function $f(\nu)$ in cases A and B, we notice that $\sigma_{B_{max}}^2(\lambda) \geq 1 = \sigma_{A_{max}}^2$ and $\sigma_{B_{min}}^2 \leq f_{max}^2(\lambda) = \sigma_{A_{min}}^2$, and hence

$$\text{cond}(\mathbf{A}_A)(\lambda) \leq \text{cond}(\mathbf{A}_B)(\lambda), \quad (34)$$

proving that the condition number is smaller when using an ISI-free filter, compared to using its square root, which has been numerically shown in [12].

IV. NUMERICAL EXAMPLE

In this section, we study the family of prototype filters with roll-off factor α , being obtained with the generator function

$$f(\nu) = \begin{cases} 1, & 0 \leq \nu \leq \frac{1-\alpha}{2K} \\ f^a\left(\frac{2K}{\alpha}[\nu - \frac{1}{2K}]\right), & \frac{1-\alpha}{2K} < \nu \leq \frac{1+\alpha}{2K} \\ -1, & \frac{1+\alpha}{2K} < \nu \leq \frac{1}{K} \end{cases}. \quad (35)$$

f^a is real-valued anti-symmetric ($f^a(x) = f^a(-x)$), and decreasing from 1 to -1 for $x \in [-1, 1]$. Therefore, $f(\nu) = -f(\frac{1}{K} - \nu)$. Hence, $f(\nu)$ can construct pulse shapes according to (20) or (28). From (33), (24) and using (35), we find that for $M\alpha \leq S(\lambda)$, $\text{cond}(\mathbf{A}_A) = \text{cond}(\mathbf{A}_B) = 1$. For $S(\lambda) \leq M\alpha$,

$$\begin{aligned} \text{cond}(\mathbf{A}_A)(\lambda) &= \frac{1}{\left|f^a\left(\frac{S(\lambda)}{\alpha M}\right)\right|}, \\ \text{cond}(\mathbf{A}_B)(\lambda) &= \frac{\left|f^a\left(\frac{S(\lambda)}{\alpha M}\right)\right|}{1 - \sqrt{1 - f^{a2}\left(\frac{S(\lambda)}{\alpha M}\right)}}. \end{aligned} \quad (36)$$

The condition number is independent of K and, based on the properties of f^a , increases with αM . As a particular example,

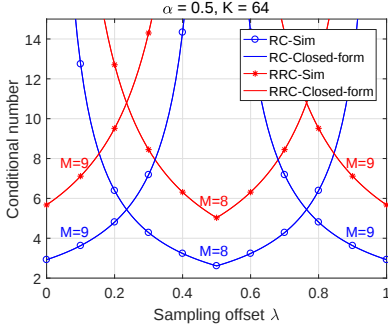


Fig. 1: Conditional number.

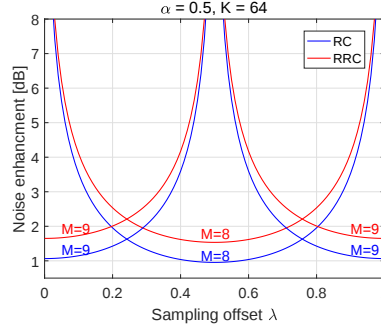
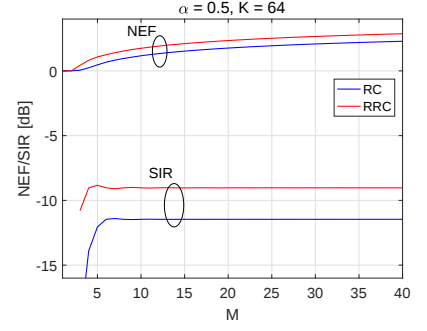


Fig. 2: Noise enhancement Factor.

Fig. 3: NEF and SIR for optimal λ .

RC and RRC use the function $f^a(x) = -\sin(\frac{\pi}{2}x)$. Replacing in (36) we get,

$$\begin{aligned} \text{cond}(\mathbf{A}_{\text{RC}})(\lambda) &= \left(\sin \left(\frac{\pi}{2} \frac{S(\lambda)}{\alpha M} \right) \right)^{-1}, \\ \text{cond}(\mathbf{A}_{\text{RRC}})(\lambda) &= \left(\tan \left(\frac{\pi}{4} \frac{S(\lambda)}{\alpha M} \right) \right)^{-1}. \end{aligned} \quad (37)$$

Fig. 1 shows the condition number of $\mathbf{A}$ for different sampling shift λ , and validates the closed-form expressions (37) numerically. As shown, $\lambda = 0$ is optimal for odd M and $\lambda = \frac{1}{2}$ for even M when K is also even. In addition, as proved in (37), using RC yields a better conditioned $\mathbf{A}$ than RRC. Furthermore, numerically obtained values for NEF as shown in Fig. 2 behave similarly as the condition number. This can be explained by the influence of the smaller singular value on the noise enhancement. In both cases, the condition number as well as the smallest singular value depend on $\left| f^a \left(\frac{S(\lambda)}{\alpha M} \right) \right| = \sin \left(\frac{\pi}{2} \frac{S(\lambda)}{\alpha M} \right)$. Considering the optimum λ , Fig. 3 illustrates NEF and SIR with different M . The proper choice of λ with respect to M preserves the trend of NEF which increases with M . On the other hand, SIR is independent of M when M is big enough. In fact, SIR approaches the interference value that can be directly obtained from $\text{SIR} = 2 \int_{\frac{1}{2K}}^{\frac{1}{2}} |H(\nu)|^2 d\nu$, which is independent of λ and K but depends on α .

V. CONCLUSION

For the waveform GFDM, the condition number of its modulation matrix is fully characterized by the adopted pulse shaping filter. In this letter, we observed that a frequency-domain shift of the frequency response of the pulse shaping filter can yield a change in the condition number. By deriving a closed-form expression of the condition number, we can find the optimal shift that minimizes the condition number for GFDM modulation. This yields a filter design that permits GFDM to have an arbitrary numbers of subcarriers and subsymbols per subcarrier, in particular power-of-two values become possible. We numerically verified the obtained closed-form expression and computed the NEF and SIR with respect to ZF and MF receivers in an additive white Gaussian noise (AWGN) channel, indicating that an optimal condition number yields also optimal NEF values.

ACKNOWLEDGMENT

The work presented in this paper has been performed in the framework of the SATURN project with contract no. 100235995 funded by the Europaischer Fonds für regionale Entwicklung (EFRE) and by the Federal Ministry of Education and Research within the programme "Twenty20 - Partnership for Innovation" under contract 03ZZ0505B - "fast wireless".

REFERENCES

- [1] P. Banelli *et al.*, "Modulation Formats and Waveforms for 5G Networks: Who Will Be the Heir of OFDM?: An overview of alternative modulation schemes for improved spectral efficiency," *IEEE Signal Processing Mag.*, vol. 31, no. 6, pp. 80–93, Nov. 2014.
- [2] N. Michailow *et al.*, "Generalized frequency division multiplexing for 5th generation cellular networks," *IEEE Trans. Commun.*, vol. 62, no. 9, pp. 3045–3061, Sep. 2014.
- [3] M. Matthé *et al.*, "Generalized frequency division multiplexing in a gabor transform setting," *IEEE Commun. Lett.*, vol. 18, no. 8, pp. 1379–1382, 2014.
- [4] H. Bölcskei and F. Hlawatsch, "Discrete Zak transforms, polyphase transforms, and applications," *IEEE Trans. Signal Process.*, vol. 45, no. 4, pp. 851–866, apr 1997.
- [5] D. W. Lin and P.-S. Wang, "On the configuration-dependent singularity of GFDM pulse-shaping filter banks," *IEEE Commun. Lett.*, vol. 20, no. 10, pp. 1975–1978, 2016.
- [6] A. Farhang *et al.*, "Low complexity GFDM receiver design: A new approach," in *2015 IEEE International Conference on Communications (ICC)*, London, UK, June 2015, pp. 4775–4780.
- [7] D. Zhang *et al.*, "Expectation Propagation for Near-Optimum Detection of MIMO-GFDM Signals," *IEEE Trans. Wireless Commun.*, vol. 15, no. 2, pp. 1045 – 1062, 2015.
- [8] V. W. S. Wong *et al.*, *Key Technologies for 5G Wireless Systems*. Cambridge University Press, 2017.
- [9] V. Klemm and A. Laub, "The singular value decomposition: Its computation and some applications," *IEEE Trans. Autom. Control.*, vol. 25, no. 2, pp. 164–176, 1980.
- [10] N. Michailow *et al.*, "Generalized frequency division multiplexing: A flexible multi-carrier modulation scheme for 5th generation cellular networks," in *Proceedings of the German microwave conference (GeMiC 12)*, Ilmenau, Germany, 2012, pp. 1–4.
- [11] X.-G. Xia, "A family of pulse-shaping filters with isi-free matched and unmatched filter properties," *IEEE Trans. Commun.*, vol. 45, no. 10, pp. 1157–1158, 1997.
- [12] M. Matthé *et al.*, "Influence of Pulse Shaping on Bit Error Rate Performance and Out of Band Radiation of Generalized Frequency Division Multiplexing," in *ICC'14 - Workshop on 5G Technologies (ICC'14 WS - 5G)*, Sydney, Australia, 2014, pp. 43–48.
- [13] R. M. Gray *et al.*, "Toeplitz and circulant matrices: A review," *Foundations and Trends® in Communications and Information Theory*, vol. 2, no. 3, pp. 155–239, 2006.

APPENDIX

Let $\mathbf{S} \in \mathbb{C}^{QL \times QL}$ be a block circulant matrix generated from diagonal matrices, such that

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_0 & \mathbf{S}_{Q-1} & \cdots & \mathbf{S}_1 \\ \vdots & \vdots & & \vdots \\ \mathbf{S}_{Q-1} & \cdots & & \mathbf{S}_0 \end{bmatrix}, \quad (38)$$

where $\mathbf{S}_q = \text{diag}\{\mathbf{v}_q\} \in \mathbb{C}^{L \times L}$, and $\mathbf{v}_q$ is the q -th column of a matrix $\mathbf{V} \in \mathbb{C}^{L \times Q}$, i.e. $\mathbf{v}_q = [\mathbf{V}]_{(:,q)}$, then [13]

$$\mathbf{S} = \mathbf{\Pi}_{L,Q}^T \mathbf{U}_{L,Q}^H \mathbf{\Lambda} \mathbf{U}_{L,Q} \mathbf{\Pi}_{L,Q}. \quad (39)$$

$$\text{where, } \mathbf{\Lambda} = \text{diag}\{\text{vec}\{\mathbf{F}_Q \mathbf{V}^T\}\}. \quad (40)$$

Using the notations $\mathbf{x}^{(i)} = \mathbf{x}[\leq n-i >_N]$ and defining the repletion matrix $\mathbf{R}_{L,Q} \in \mathbb{R}^{LQ \times L}$, $\mathbf{R}_{L,Q} = \mathbf{1}_L \otimes \mathbf{I}_Q$, the transmitted GFDM block in (1) can be expressed in the following vector form,

$$\begin{aligned} \mathbf{x} &= \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} d_{k,m} \text{diag}\{\mathbf{g}^{(mK)}\} \mathbf{R}_{M,K} [\mathbf{F}_K^H]_{(:,k)}, \\ &= \sum_{m=0}^{M-1} \sqrt{K} \text{diag}\{\mathbf{g}^{(mK)}\} \mathbf{R}_{M,K} \frac{1}{\sqrt{K}} \mathbf{F}_K^H \mathbf{d}_m \\ &= \mathbf{S}^{(M)} \mathbf{U}_{M,K}^H \text{vec}\{\mathbf{D}\} = \mathbf{A} \cdot \text{vec}\{\mathbf{D}\}. \end{aligned} \quad (41)$$

Here, $\mathbf{A} = \mathbf{S}^{(M)} \mathbf{U}_{M,K}^H$ and

$$\frac{\mathbf{S}^{(M)}}{\sqrt{K}} = [\text{diag}\{\mathbf{g}^{(0K)}\} \mathbf{R}_{M,K}, \dots, \text{diag}\{\mathbf{g}^{((M-1)K)}\} \mathbf{R}_{M,K}] \quad (42)$$

is block circulant matrix as in (38), with $\mathbf{S}_m^{(M)} = \text{diag}\{\sqrt{K} [\mathbf{V}_{K,M}^{(g)}]_{(:,m)}\}$. From (39), we get

$$\mathbf{S}^{(M)} = \mathbf{\Pi}_{K,M}^T \mathbf{U}_{K,M}^H \mathbf{\Lambda}^{(g)} \mathbf{U}_{K,M} \mathbf{\Pi}_{K,M}, \quad (43)$$

$$\mathbf{\Lambda}^{(g)} = \sqrt{K} \text{diag}\left\{\text{vec}\left\{\mathbf{F}_M \left(\mathbf{V}_{K,M}^{(g)}\right)^T\right\}\right\}. \quad (44)$$

As a result we get $\mathbf{A}$ defined in (6).

The N -FFT of (1) results in

$$\tilde{\mathbf{x}}[n] = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} d_{k,m} \tilde{g}[\leq n-kM >_N] e^{-j2\pi \frac{m}{M} n}. \quad (45)$$

Following similar steps we get

$$\tilde{\mathbf{x}} = \mathbf{F}_N \mathbf{A} \cdot \text{vec}\{\mathbf{D}\}. \quad (46)$$

Here $\mathbf{F}_N \mathbf{A} = \mathbf{S}^{(K)} \mathbf{U}_{K,M} \mathbf{\Pi}_{M,K}$ and

$$\frac{\mathbf{S}^{(K)}}{\sqrt{M}} = [\text{diag}\{\tilde{\mathbf{g}}^{(0M)}\} \mathbf{R}_{K,M}, \dots, \text{diag}\{\tilde{\mathbf{g}}^{((K-1)M)}\} \mathbf{R}_{K,M}]. \quad (47)$$

By replacing $\mathbf{S}^{(K)}$ as in (44), then

$$\frac{1}{\sqrt{N}} \mathbf{F}_N \mathbf{A} = \mathbf{\Pi}_{M,K}^T \mathbf{U}_{M,K}^H \mathbf{\Lambda}^{(\tilde{g})} \mathbf{U}_{M,K} \mathbf{\Pi}_{M,K} \mathbf{U}_{K,M} \mathbf{\Pi}_{M,K}. \quad (48)$$

$$\mathbf{\Lambda}^{(\tilde{g})} = \frac{1}{\sqrt{K}} \text{diag}\left\{\text{vec}\left\{\mathbf{F}_K \left(\mathbf{V}_{M,K}^{(\tilde{g})}\right)^T\right\}\right\}. \quad (49)$$

And finally by multiplying whit $\frac{1}{\sqrt{N}} \mathbf{F}_N$, we get (7).